

Like eddy viscosity, α_t is not a fluid property but depends on the state of turbulence. In fact the Reynolds analogy between heat and momentum transport suggests

$$\alpha_t = \frac{\nu_t}{\sigma_t} \quad (6.30)$$

The denominator σ_t is called turbulent Prandtl number. Experiments have shown that σ_t varies very little across the flow. Many models make use of σ_t as a constant. For the flow of air a value of 0.9 may be chosen.

Finally the transport equation (6.27) may be written as

$$\begin{aligned} \bar{u} \frac{\partial \bar{T}}{\partial x} + \bar{v} \frac{\partial \bar{T}}{\partial y} + \bar{w} \frac{\partial \bar{T}}{\partial z} &= \frac{\partial}{\partial x} \left[(\alpha + \alpha_t) \frac{\partial \bar{T}}{\partial x} \right] \\ &+ \frac{\partial}{\partial y} \left[(\alpha + \alpha_t) \frac{\partial \bar{T}}{\partial y} \right] + \frac{\partial}{\partial z} \left[(\alpha + \alpha_t) \frac{\partial \bar{T}}{\partial z} \right] \end{aligned} \quad (6.31)$$

6.4 PRANDTL'S MIXING LENGTH

In the case of turbulent flow, velocity, pressure and temperature at a fixed point in space do not remain constant with time but undergo very irregular fluctuations of high frequency. The velocity of the fluid in turbulent flow can be represented by two components

- (i) mean motion
- (ii) fluctuating component (or eddying motion)

The velocity of fluid is given by

$$u = \bar{u} + u', \quad v = \bar{v} + v', \quad w = \bar{w} + w'$$

Similarly,

$$p = \bar{p} + p', \quad T = \bar{T} + T'$$

However, we must always consider that the fluctuating components do not bring about the bulk displacement of a fluid element. Consider continuity equation in two dimensions: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$. If we substitute $u = \bar{u} + u'$ and $v = \bar{v} + v'$ and then perform time averaging, we shall obtain

$$\frac{\partial(\overline{\bar{u} + u'})}{\partial x} + \frac{\partial(\overline{\bar{v} + v'})}{\partial y} = 0$$

or

$$\left[\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} \right] + \left[\frac{\partial \overline{u'}}{\partial x} + \frac{\partial \overline{v'}}{\partial y} \right] = 0 \quad (6.32)$$

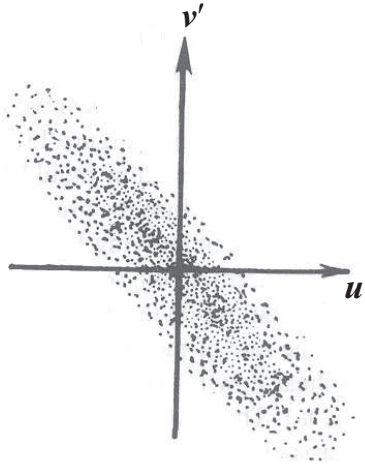


Figure 6.5 Each dot represents a $u'v'$ pair at a given time

But we have seen that

$$\overline{u'} = \overline{v'} = \frac{\partial \overline{u'}}{\partial x} = \frac{\partial \overline{v'}}{\partial y} = 0$$

Invoking

$$\frac{\partial \overline{u'}}{\partial x} = \frac{\partial \overline{v'}}{\partial y} = 0$$

in Eq. (6.32) we get $\frac{\partial \overline{u'}}{\partial x} + \frac{\partial \overline{v'}}{\partial y} = 0$ which yields

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} = 0$$

we can also write

$$\frac{\partial u'}{\partial x} = -\frac{\partial v'}{\partial y} \quad (6.33)$$

If we consider momentum exchange between two adjacent layers, then on the basis of above equation, it is postulated that if at any instant there is an increase in u' in the x direction it will be followed by an increase in v' in the negative y direction. In other words $\overline{u'v'}$ is nonzero and negative. This is discerned in Fig. 6.5 which shows a cloud of data points (sometimes called a scatter plot). The dots represent the instantaneous values of $u'v'$ pair at different times.

For unit area of the plane PP (Fig. 6.6) the instantaneous turbulent mass transport rate across the plane is $\rho v'$. Associated with this mass transport is a change in the x component of velocity u' . The net momentum flux per unit area, in the x direction, represents the turbulent shear stress at the plane PP (Fig. 6.6) which is $\rho u'v'$. When a turbulent lump movement is in the upward direction ($v' > 0$), it

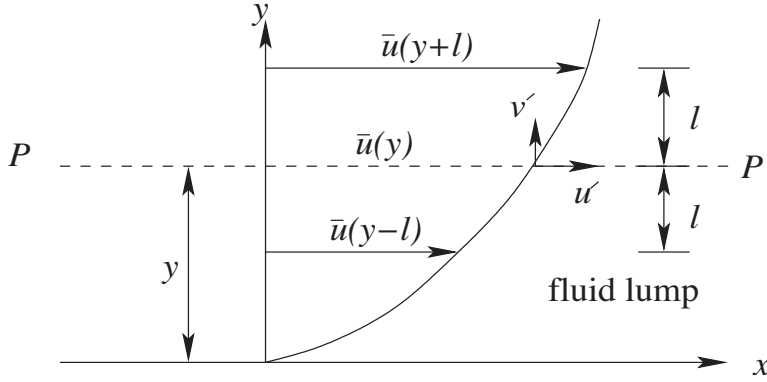


Figure 6.6 Mixing length hypothesis

enters a region of higher $\bar{u}$ and is therefore likely to effect a slowing down fluctuation in u' , that is $u' < 0$. A similar argument can be made for $v' < 0$, so that the average turbulent shear stress will be given as

$$\tau_t = -\rho \overline{u'v'} \quad (6.34)$$

Again, let us imagine a turbulent fluid lump which is located a distance l above or below the plane PP. These lumps of fluid move back and forth across the plane and give rise to the eddy or turbulent-shear-stress effect.

At $(y + l)$, the velocity would be

$$\bar{u}(y + l) = \bar{u}(y) + l \frac{\partial \bar{u}}{\partial y} \quad (6.35)$$

while at $(y - l)$, we can write

$$\bar{u}(y - l) = \bar{u}(y) - l \frac{\partial \bar{u}}{\partial y} \quad (6.36)$$

Prandtl postulated that the turbulent fluctuation is proportional to the mean of the above two quantities

$$|u'| \approx \frac{1}{2}(|\Delta u_1| + |\Delta u_2|)$$

or

$$u' \approx l \frac{\partial \bar{u}}{\partial y} = C_1 l \frac{\partial \bar{u}}{\partial y} \quad (6.37)$$

Here l is Prandtl's mixing length. This is analogous to mean free path (average distance a particle travels between collisions) in molecular transport problems. Prandtl also postulated that v' is of the same order of magnitude as u' so that

$$v' = C_2 l \frac{\partial \bar{u}}{\partial y} \quad (6.38)$$

Now, the turbulent shear stress could be written as

$$\tau_t = -\overline{\rho u'v'} = \rho l^2 \left(\frac{\partial \bar{u}}{\partial y} \right)^2 = \rho \nu_t \frac{\partial \bar{u}}{\partial y} \quad (6.39)$$

The constants C_1 and C_2 can be included in still unknown mixing length, l . The eddy viscosity thus becomes,

$$\nu_t = l^2 \left| \frac{\partial \bar{u}}{\partial y} \right| \quad (6.40)$$

Consider the boundary layer on a flat plate. The shear stress due to laminar flow is given by $\tau_l = \mu \frac{\partial \bar{u}}{\partial y}$. The total shear stress is

$$\tau = \tau_l + \tau_t = \rho \nu \frac{\partial \bar{u}}{\partial y} + \rho \nu_t \frac{\partial \bar{u}}{\partial y} \quad (6.41)$$

or

$$\tau = \rho(\nu + \nu_t) \frac{\partial \bar{u}}{\partial y} \quad (6.42)$$

The turbulent viscosity, ν_t in Eq. (6.42) can be determined from Eq. (6.40). However, our problem is still not resolved. How do we determine the value of l , the mixing length? Several correlations, using experimental results for τ_t have been proposed to determine l . In the regime of isotropic turbulence, the most widely used value of mixing length is

$$l = \chi y \quad (6.43)$$

where y is the distance from the wall and χ is known as von Karman constant. Experiments have shown that χ is approximately equal to 0.4 (Tennekes and Lumley [1]).

6.5 UNIVERSAL VELOCITY PROFILE (ON FLAT PLATE)

The flow fields that are dominated by high frequency oscillations (small time periods), are isotropic and the mean velocity profiles in such flow fields are universal. In this section we shall discuss the universal velocity profile.

A very thin layer next to the wall behaves like a near wall region of laminar flow. The layer is known as laminar sub-layer and its velocities are such that the viscous forces dominate over the inertia forces (Fig. 6.7). No turbulence exists in it. We know that inertial effects are insignificant in the near wall region and can write

$$\nu \frac{\partial^2 \bar{u}}{\partial y^2} - \frac{\partial \overline{u'v'}}{\partial y} = 0 \quad (6.44)$$

which can be integrated as

$$\nu \frac{\partial \bar{u}}{\partial y} - \overline{u'v'} = \text{constant} \quad (6.45)$$

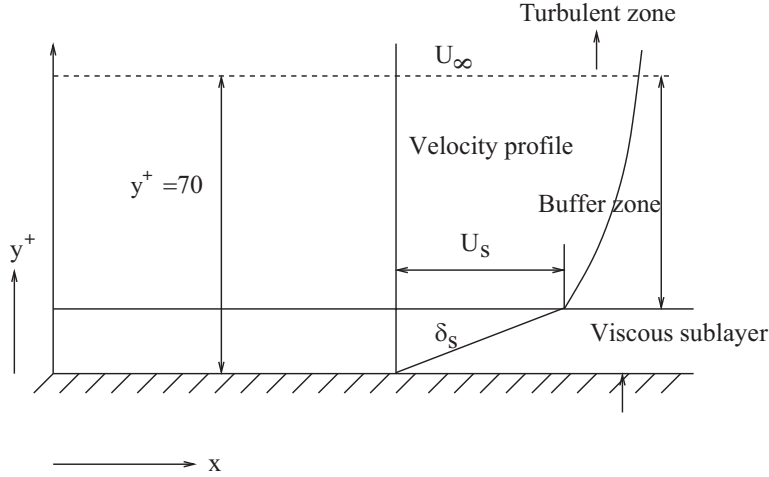


Figure 6.7 Different zones of a turbulent flow past a wall

Again, as we know that the fluctuating components vanish near the wall, the shear stress on the wall is purely viscous and it follows that:

$$\nu \frac{\partial \bar{u}}{\partial y} = \frac{\tau_w}{\rho} \quad (6.46)$$

or

$$\frac{\bar{u} - 0}{y - 0} = \frac{\tau_w}{\rho \nu} = \frac{u_\tau^2}{\nu} \quad \text{where} \quad u_\tau = \sqrt{\frac{\tau_w}{\rho}} \quad (6.47)$$

The quantity u_τ is known as friction velocity, given by $\tau_w = \rho u_\tau^2$. From Eq. (6.47), it is possible to write

$$y \frac{u_\tau}{\nu} = \frac{\bar{u}}{u_\tau} \quad (6.48)$$

Hence a nondimensional coordinate may be defined as $y^+ = y u_\tau / \nu$ and we write down the variation of the nondimensional velocity within the sublayer as

$$\bar{u}^+ = y^+ \quad (6.49)$$

In the turbulent zone, the turbulent shear stress from Prandtl's mixing length model can be written as

$$\tau_t = \rho l^2 \left[\frac{\partial \bar{u}}{\partial y} \right]^2 \quad (6.50)$$

Where l is the mixing length $= \chi y$ and χ is a von Karman constant we write

$$\frac{\tau_t}{\rho} = \chi^2 y^2 \left[\frac{\partial \bar{u}}{\partial y} \right]^2 \quad (6.51)$$

The shear stress in the turbulent region dictates the velocity U_s at the edge of the sublayer (Fig.6.7). The wall shear stress is estimated as $\tau_w = \mu U_s / \delta_s$.

$$\left[\frac{\partial \bar{u}}{\partial y} \right]^2 = \left[\frac{u_\tau}{\chi y} \right]^2 \quad [\text{since } \tau_t = \tau_w] \quad (6.52)$$

$$\bar{u} = \frac{u_\tau}{\chi} \ln y + \text{constant} \quad (6.53)$$

or

$$\frac{\bar{u}}{u_\tau} = \frac{1}{\chi} [\ln y - \ln y_0] \quad \left[y_0 \text{ being very small} = \frac{\beta \nu}{u_\tau} \right] \quad (6.54)$$

or

$$\frac{\bar{u}}{u_\tau} = \frac{1}{\chi} \left[\ln \frac{y u_\tau}{\nu} - \ln \beta \right] \quad (6.55)$$

or

$$\bar{u}^+ = A_1 \ln y^+ + D_1 \quad \left[\text{put } A_1 = \frac{1}{\chi} \right] \quad (6.56)$$

These constants were determined from experiments. For smooth ducts, A_1 has been observed as $A_1 = 2.5$ and $D_1 = 5.5$ or

$$\bar{u}^+ = 2.5 \ln y^+ + 5.5 \quad (6.57)$$

or $\bar{u}^+ = 2.5 \ln [Ey^+]$ where $E = 9.0$ for smooth walls.

Finally the log-law is defined for the turbulent zone as

$$\bar{u}^+ = \frac{1}{\chi} \ln [Ey^+] \quad (6.58)$$

The coefficient χ is known as the von Karman constant.

The location where log profile and linear profile meet can be calculated as:

$$\underbrace{\bar{u}^+ = \frac{1}{\chi} \ln [Ey^+]}_{\text{log-law}} \quad (6.59)$$

or

$$y^+ = \frac{1}{\chi} \ln [Ey^+] \quad (\text{from Eq. 6.49})$$

or

$$y^+ = 11.63 \quad (6.60)$$

As such, boundary layer thickness is around $1500 y^+$.